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Abel-Gončarov Polynomial Expansions

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1. INTRODUCTION

The *Gončarov polynomials* are defined by

$$G_0(z) = 1$$

and

$$G_n(z; z_0, z_1, ..., z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, z_1, ..., z_{k-1}), \qquad (n = 1, 2, 3, ...)$$

where $\{z_k\}_{k=0}^{\infty}$ is an arbitrary sequence of complex numbers. These polynomials are biorthogonal to the linear functionals

$$L_n(f) = f^{(n)}(z_n); \qquad (n = 0, 1, 2, ...)$$

that is,

$$L_n[G_k(z; z_0, z_1, ..., z_{k-1})] = \delta_{nk}.$$

The question of expansion of functions, analytic in a neighborhood of 0, in the polynomial series

$$\sum_{n=0}^{\infty} L_n(f)\, G_n(z; z_0, ..., z_{n-1}) \tag{1.1}$$

was first considered by Abel and Goncarov [13]. This expansion can be shown to converge to the function f in a number of interesting cases.

Suppose that f is an entire function of exponential type less than 1. Let

$$H_n = \max |G_n(0; w_0, ..., w_{n-1})|,$$

where the maximum is taken over all sequences $\{w_k\}_{k=0}^{n-1}$ whose terms lie in the disk $|z| \leqslant 1$, and set

$$P = \lim\sup H_n^{1/n}.$$

The constant P lies between 1.355 and 1.378; the series (1.1) converges uniformly on bounded subsets of the plane to the function f for each sequence $\{z_n\}_{n=0}^{\infty}$ such that $|z_n| \leqslant 1/P$. This was proved in 1954 by M. A. Evgrafov [10]. Using different methods, J. D. Buckholtz [2] also obtained the expansion and proved additionally that

$$P = \lim_{n\to\infty} H_n^{1/n} = \sup_{1\leqslant n<\infty} H_n^{1/n}.$$

The expansion (1.1) also holds for functions with finite radii of convergence. The first result in this direction is due to M. M. Dragilev [6]: if f is analytic in $\mathscr{U} = \{z: |z| < 1\}$, $0 < R < 1$, and $|z_n| \leqslant R/P(n+1)$, $n = 0, 1, 2,...$, then (1.1) converges uniformly to f on compact subsets of $\mathscr{U}$.

The *remainder polynomials* are defined recursively by

$$B_0(z) = 1$$

and

$$B_n(z; z_0, z_1,..., z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, z_1,..., z_{k-1}), \quad (n = 1, 2, 3,...).$$

In analogy to787Gončarov polynomials and derivatives, the remainder polynomials have been useful in investigating zeros of remainders of power series. For a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in a neighborhood of 0, let $\mathscr{S}$ denote the operator which transforms f into

$$\mathscr{S}f(z) = \sum_{n=1}^{\infty} a_n z^{n-1},$$

and let $\mathscr{S}^k$ be defined as the kth successive iterate of $\mathscr{S}$. The function $\mathscr{S}^k f(z)$ is sometimes called the k-th *normalized remainder* of the power series for f. The remainder polynomials are biorthogonal to the linear functionals

$$l_n(f) = \mathscr{S}^n f(z_n)$$

and hence lead us to the Abel–Gončarov series

$$\sum_{n=0}^{\infty} l_n(f)\, B_n(z; z_0,..., z_{n-1}). \tag{1.2}$$

M. Pommiez [18] considered this expansion for functions analytic in $\mathscr{U}$, with the interpolation points $\{z_n\}$ lying in a closed disk of radius r, $r \leqslant .539$, and showed that (1.2) converges uniformly to f on compact subsets of $\mathscr{U}$. This result, however, has been sharpened to best possible form. Let

$$h_n = \max | B_n(0; w_0, ..., w_{n-1})|,$$

where $| w_j | \leqslant 1$ for $0 \leqslant j \leqslant n - 1$, and set

$$P = \limsup h_n^{1/n}.$$

M. M. Dragilev [7] and the first author [3] proved independently that for functions f analytic in $\mathscr{U}$ and sequences $\{z_n\}$ satisfying $| z_n | \leqslant r < 1/P$, the series (1.2) converges uniformly to f on compact subsets of $\mathscr{U}$. Note that $1/P$ lies between .549 and .561, giving a better estimate for Pommiez's constant.

In the present paper, we consider expansions of analytic functions in series of certain polynomials which specialize to both the Gončarov and remainder polynomials. The corresponding operator is sufficiently general to deduce the expansions (1.1) and (1.2) as special cases, and also to obtain similar results for entire functions of arbitrary order and type.

Let $\{d_n\}_{n=1}^{\infty}$ denote a nondecreasing sequence of positive numbers and let $\mathscr{D}$ denote the linear operator defined by $\mathscr{D}(z^n) = d_n z^{n-1}$. Thus if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then $\mathscr{D}$ transforms f into

$$\mathscr{D}f(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}.$$

The operator $\mathscr{D}$ is sometimes called the *Gel'fond–Leont'ev* [12] derivative of f, and is easily seen to correspond to the ordinary derivative D when $d_n = n$ and to $\mathscr{S}$ when $d_n = 1$. The operators $\mathscr{D}^n$ $(n = 1, 2, 3,...)$ are the successive iterates of $\mathscr{D}$. If we let $e_0 = d_0 = 1$ and $e_n = (d_1 d_2 \cdots d_n)^{-1}$, for $n \geqslant 1$, then

$$\mathscr{D}^n f(z) = \sum_{k=n}^{\infty} \frac{e_{k-n}}{e_k} a_k z^{k-n}, \tag{1.3}$$

and

$$e_n \mathscr{D}^n f(0) = a_n \tag{1.4}$$

for each n. We observe also that the role played by the sequence e_n is that if $p_n(z) = e_n z^n$, then $\mathscr{D}(p_n) = p_{n-1}$.

We define the growth measure *E-type* as follows: if $\{R_n\}_{n=0}^{\infty}$ is a nondecreasing sequence of positive numbers, then the *E*-type of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is

$$\tau_E(f) = \limsup | a_n R_1 R_2 \cdots R_n |^{1/n}.$$

The function E defined by

$$E(z) = \sum_{k=0} z^k/(R_0 R_1 \cdots R_k) \qquad (R_0 = 1)$$

has E-type 1 and radius of convergence

$$c(E) = \sup_{1 \leqslant n < \infty} R_n .$$

If $c(E) < \infty$, then $\tau_E(f) = c(E)/c(f)$. If $c(E)$ is infinite, then E-type corresponds to a growth measure introduced by L. Nachbin [1, 16]. In particular, if $R_n \equiv n$, then E-type agrees with exponential type. The conditions $c(E) = \infty$ and $\tau_E(f) < \infty$ clearly imply that f is entire.

In the sequel, we will always require that the sequences $\{R_n\}$ and $\{d_n\}$ satisfy the following conditions:

(a) $\{R_{n+1}/R_n\}_{n=1}^{\infty}$ has limit 1;

(b) $\{d_{n+1}/d_n\}_{n=1}^{\infty}$ is nonincreasing and has limit 1.

The polynomials $Q_n(z; z_0, z_1, \ldots, z_{n-1})$ are defined by

$$Q_0(z) = 1$$

and

$$Q_n(z; z_0, z_1, \ldots, z_{n-1}) = e_n z^n - \sum_{k=0}^{n-1} e_{n-k} z_k^{n-k} Q_k(z; z_0, z_1, \ldots, z_{k-1}),$$
$$(n = 1, 2, 3, \ldots)$$

where $\{z_k\}_{k=0}^{\infty}$ is a sequence of complex numbers. In Section 2, we prove that these polynomials are biorthogonal to the linear functionals

$$\mathscr{L}_n(f) = \mathscr{D}^n f(z_n).$$

The Q_n polynomials reduce to the Gončarov polynomials if $d_n \equiv n$ and to the remainder polynomials if $d_n \equiv 1$. Let

$$H_n = \max | Q_n(0; w_0, \ldots, w_{n-1})|$$

where the maximum is taken over all sequences $\{w_k\}_{k=0}^{n-1}$ whose terms lie in $\mathscr{U}$ (we drop the previous notational convention that H_n refers only to the Gončarov polynomials) and let

$$W(\mathscr{D}) = \{ \sup_{1 \leqslant n < \infty} H_n^{1/n} \}^{-1}.$$

$W(\mathscr{D})$ is called the *Whittaker constant* belonging to the operator $\mathscr{D}$. Clearly

$W(D) = 1/P$ and $W(\mathscr{S}) = 1/p$. In [3], J. D. Buckholtz and J. L. Frank characterized the sequence $\{H_n\}$ and proved that

$$W(\mathscr{D}) = \{\lim_{n\to\infty} H_n^{1/n}\}^{-1} = \{\sup_{1\leqslant n<\infty} H_n^{1/n}\}^{-1} \tag{1.6}$$

and

$$d_1/2 \leqslant W(\mathscr{D}) < d_1 . \tag{1.7}$$

We shall usually abbreviate $W(\mathscr{D})$ to W when no confusion is likely as to the operator under consideration.

Our main result here is the following

THEOREM A. *If* $\tau_E(f) \leqslant 1$, $0 < c < W$, *and the sequence* $\{z_n\}$ *satisfies* $|z_n| \leqslant cR_{n+1}/d_{n+1}$, *then*

$$f(z) = \sum_{n=0}^{\infty} \mathscr{D}^n f(z_n)\, Q_n(z; z_0, ..., z_{n-1}) \tag{1.8}$$

with uniform convergence on compact subsets of $|z| < c(f)$.

If $R_n = d_n \equiv n$, then (1.8) reduces to (1.1) for the case in which f is entire with exponential type less than or equal to 1. If $R_n \equiv 1$ and $d_n \equiv n$, we obtain the expansion (1.1) for functions analytic in $\mathscr{U}$ and sequences $\{z_n\}$ satisfying $|z_n| \leqslant 1/P(n+1)$. If $R_n = d_n \equiv 1$, then (1.8) reduces to (1.2) for f analytic in $\mathscr{U}$.

A natural question associated with (1.8) is whether the constant W can be replaced by a larger number in Theorem A. That W is best possible is an easy consequence of the following theorem due to Buckholtz and Frank [4].

THEOREM B. *There exists a function F, of E-type* 1, *such that if* $c > W$ *then* $\mathscr{D}^n F$ *has a zero in* $|z| \leqslant cR_{n+1}/d_{n+1}$ *for all but finitely many n.*

Finally, we mention two related expansion problems associated with the class $E_{\rho\tau}$ of entire functions of order ρ, $0 < \rho < \infty$, and positive type not exceeding τ, $\tau < \infty$. Let γ denote the supremum of numbers c such that if $f \in E_{\rho\tau}$ and $(n+1)^{1-(1/\rho)} |z_n| \leqslant c(\rho\tau)^{1/\rho}$, for $n = 0, 1, 2, ...$, then

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_n)\, G_n(z; z_0, ..., z_{n-1}) \tag{1.9}$$

uniformly on bounded sets. In 1962, M. M. Džrbašjan [9] proved that $\gamma \geqslant \log 2$. For the sequence $R_n = (n/\rho\tau)^{1/\rho}$, $f \in E_{\rho\tau}$ implies $\tau_E(f) \leqslant 1$. If one takes $d_n = n$, then Theorem A yields (1.9) for any sequence $\{z_n\}$ such that $(n+1)^{1-(1/p)} |z_n| \leqslant W(D)\, R/(\rho\tau)^1/\rho$, where $0 < R < 1$, and it follows that $\gamma \geqslant W(D)$. In view of Theorem B, $\gamma = W(D)$.

Let γ' denote the supremum of numbers c such that $f \in E_{\rho\tau}$ and $|z_n| \leqslant c(n/\rho\tau)^{1/\rho}$ imply

$$f(z) = \sum_{n=0}^{\infty} \mathscr{S}^n f(z_n)\, B_n(z; z_0, \ldots, z_{n-1}) \tag{1.10}$$

uniformly on bounded sets. Here, we take $R_n = (n/\rho\tau)^{1/\rho}$ and $d_n = 1$. Theorem A implies (1.10) for $|z_n| \leqslant W(\mathscr{S})\, R(n/\rho\tau)^{1/\rho}$, where $0 < R < 1$, and therefore $\gamma' \geqslant W(\mathscr{S})$. The reverse inequality $\gamma' \leqslant W(\mathscr{S})$ is obtained from Theorem B.

2. The Polynomials Q_n

All of the basic properties of these polynomials were developed in [3]. listed below are the properties we shall need.

Lemma 2.1. *The following identities hold:*

$$Q_n(\lambda z; \lambda z_0, \ldots, \lambda z_{n-1}) = \lambda^n Q_n(z; z_0, \ldots, z_{n-1}), \tag{2.1}$$

where λ is complex,

$$Q_n(z_0; z_0, \ldots, z_{n-1}) = 0, \qquad n \geqslant 1, \tag{2.2}$$

$$\mathscr{D}^k Q_n(z; z_0, \ldots, z_{n-1}) = Q_{n-k}(z; z_k, \ldots, z_{n-1}), \qquad 0 \leqslant k \leqslant n \tag{2.3}$$

$$\mathscr{D}^k Q_n(z_k; z_0, \ldots, z_{n-1}) = \delta_{nk}, \tag{2.4}$$

$$Q_n(z; z_0, \ldots, z_{n-1}) = \sum_{k=0}^{n} Q_{n-k}(w_k; z_k, \ldots, z_{n-1})\, Q_k(z; w_0, \ldots, w_{k-1}), \tag{2.5}$$

for arbitrary $\{z_k\}$ and $\{w_k\}$.

Note that Eq. (2.4) is the biorthogonality condition of 1. If, in (2.5), we take $w_k = 0$, $0 \leqslant k \leqslant n$, and use the fact that $Q_k(z; 0, \ldots, 0) = e_k z^k$, we obtain the useful identity

$$Q_n(z; z_0, \ldots, z_{n-1}) = \sum_{k=0}^{n} Q_{n-k}(0; z_k, \ldots, z_{n-1})\, e_k z^k. \tag{2.6}$$

The following result is our basic tool in obtaining the expansion (1.8).

Lemma 2.2. *Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $c(f) > 0$,* *let n be a positive integer and suppose that the complex numbers* $z_0, z_1, \ldots, z_{n-1}$ *lie in* $|z| < c(f)$. *Then*

$$f(z) = \sum_{k=0}^{n-1} \mathscr{D}^k f(z_k)\, Q_k(z; z_0, \ldots, z_{k-1})$$

$$+ \sum_{k=n}^{\infty} \mathscr{D}^k f(0)\, Q_k(z; z_0, \ldots, z_{n-1}, 0, \ldots, 0). \tag{2.7}$$

Proof. The condition that $\{d_{n+1}/d_n\}$ have limit 1 implies that

$$\lim_{n\to\infty} d_n^{1/n} = 1.$$

Therefore

$$\mathscr{D}f(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}$$

has radius of convergence $c(f)$. From (1.3) and (1.4) we have

$$\mathscr{D}^k f(z_k) = \sum_{m=0}^{\infty} \mathscr{D}^{k+m} f(0)\, e_m z_k{}^m,$$

for $0 \leqslant k \leqslant n-1$. Substituting this expression into

$$\sum_{k=0}^{n-1} \mathscr{D}^k f(z_k)\, Q_k(z; z_0, \ldots, z_{k-1})$$

and interchanging to the order of summation, one obtains an equation equivalent to (2.7). This completes the proof.

3. Matrix Transformations on the Space $\mathscr{A}_r$

We denote by $\mathscr{A}_r (r > 0)$ the complex vector space of functions analytic in the disk $\mathscr{U}_r = \{z: |z| < r\}$, with the topology of uniform convergence on compact subsets of $\mathscr{U}_r$. The topology of $\mathscr{A}_r$ can also be defined by the seminorms

$$\|f\|_\rho = \sum_{n=0}^{\infty} |a_n| \rho^n, \qquad \rho < r, \tag{3.1}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$. It is well-known that $\mathscr{A}_r$ is a Fréchet space.

For our purposes, it is convenient to move from the function space $\mathscr{A}_r$ to its equivalent sequence space by mapping $f(z) = \sum_{k=0}^{\infty} a_k z^k$ onto the sequence $(a_0, a_1, a_2, \ldots)$. We also adopt the notational convention that the sequence space [14], topologized by the seminorms (3.1), be denoted by the same symbol $\mathfrak{A}_r$.

Let M be an infinite complex matrix and let $f = (f_0, f_1, f_2, ...) \in \mathfrak{A}_r$. Then Mf is the sequence whose n-th term is given by

$$(Mf)_n = \sum_{k=0}^{\infty} M_{nk} f_k .$$

We say that M maps $\mathfrak{A}_r$ into $\mathfrak{A}_r$ provided that $f \in \mathfrak{A}_r$ implies $Mf \in \mathfrak{A}_r$.

In case M is a homeomorphism, a certain number of expansion theorems are immediate. In fact, if $\{\sigma_n\}$ is any *basis* in $\mathfrak{A}_r$, then the images $\{M\sigma_n\}$ also form a basis. Of particular interest ([1, 20]) is the case in which M is upper triangular ($M_{jk} = 0$ whenever $j > k$) and $\sigma_n(z) = z^n$. Here, the images $M\sigma_n$ are always polynomials.

The following theorem of M. G. Haplanov [11] characterizes the matrix transformations of $\mathfrak{A}_r$ into $\mathfrak{A}_r$.

THEOREM 3.1. *The matrix M maps $\mathfrak{A}_r$ into $\mathfrak{A}_r$ if and only if for each $q > r^{-1}$ there exists a number $b < r$ such that*

$$| M_{jn} | < 0(1)\, q^j b^n \tag{3.2}$$

for all j and n.

By placing fairly restrictive conditions on the matrix M, one can extend Haplanov's result to the case in which r is allowed to assume values arbitrarily large.

THEOREM 3.2 (Dragilev). *Let M be an infinite upper triangular matrix such that $M_{kk} = 1$, for $0 \leqslant k < \infty$, let N denote the unique upper triangular inverse of M, and let R be a positive number. A necessary and sufficient condition that either M or N map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for each $r > R$ is that*

$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} | M_{jn} | R^{j-n}\}^{1/n} \leqslant 1 \tag{3.3}$$

and

$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} N_{jn} | R^{j-n}\}^{1/n} \leqslant 1 \tag{3.4}$$

Proof. This result and Theorem 3.3 below are stated in [6], without accompanying proofs. For completeness (and the fact that Dragilev's work is available only in Russian) we will provide proofs.

The sufficiency is straightforward and we omit the details. For the necessity, suppose, say, that M maps $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for all $r > R$. If

$r > R$ and $q > r^{-1}$, then by Theorem 3.1 there is a $b < r$ and a constant $K > 0$ such that

$$|M_{jn}| < Kq^j b^n$$

for all j and n. For $0 \leqslant j \leqslant n$ we have

$$\begin{aligned} |M_{jn}| R^{j-n} &\leqslant K(qr)^j (b/r)^n (r/R)^{n-j} \\ &\leqslant K(qr)^n (r/R)^n, \end{aligned}$$

and therefore

$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} |M_{jn}| R^{j-n}\}^{1/n} \leqslant (qr)(r/R).$$

Inequality (3.3) follows from noting that we can make the quantities (r/R) and (qr) arbitrarily close to 1.

There remains to show that (3.4) is satisfied. We begin by noting that the condition (3.2) implies that M is continuous. If T denotes the inverse mapping of M, the open mapping theorem implies that T is continuous. By a theorem of Köthe and Toeplitz [17], T is representable by a matrix C. For $f \in \mathfrak{A}_r$, we have

$$\begin{aligned} [(CM)f]_m &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} M_{nk} f_k \\ &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} C_{mn} M_{nk} f_k \\ &= \sum_{n=0}^{\infty} C_{mn} (Mf)_n \\ &= [C(Mf)]_m, \end{aligned}$$

there being no difficulty in interchanging the order of summation. Therefore $(CM)f = C(Mf) = T(Mf) = f$, and it follows that $C = N$. Since C maps $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for $r > R$, we obtain (3.4) by applying the first part of the proof to the matrix N.

To obtain the "either-or" statement in the theorem, one simply interchanges the symbols M and N. This observation also leads to the following corollary.

COROLLARY 3.1. *The matrices M and N either simultaneously map $\mathfrak{A}_r$ one-to-one and onto itself for each $r > R$, or neither does.*

In applications of Theorem 3.2, it is not necessary to prove (3.3) and (3.4) directly. Let M and N be defined as in Theorem 3.2 and let M' and N'

be reciprocal upper triangular matrices which satisfy $M'_{kk} = N'_{kk} = 1$, $0 \leqslant k < \infty$. If $r > 0$ and $0 \leqslant j < \infty$, define

$$\mu_j(r) = \sum_{n=j}^{\infty} | N_{jn} - N'_{jn} | r^{j-n}.$$

THEOREM 3.3. *Let $R > 0$ and suppose that each of M' and N' map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for all $r > R$. Suppose also that* $| M'_{jn} | r^{j-n} = 0(1)$, *for all n and j $r > R$, and that* $\lim_{j\to\infty} \mu_j(r) = 0$ *uniformly in* (R, ∞). *Then each of M and N map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for all $r > R$.*

Proof. Since

$$N = (I + (N - N')\, M')\, N',$$

where I denotes the identity matrix, it suffices to show that the matrix $B = I + (N - N')\, M'$ maps $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$. Let $C = (N' - N)\, M'$, so that $B = I - C$, and for $r > R$ define the matrix $C^{(r)}$ by $C^{(r)}_{jk} = C_{jk} r^{j-k}$.

We begin by showing that $C^{(r)}$ is a compact operator on l_1, with range in l_1, for each $r > R$. If $R < r_1 < r$, the bound $| M'_{jn} | r_1^{j-n} = O(1)$ implies that

$$\begin{aligned} | C^{(r)}_{jn} | &\leqslant \sum_{k=j}^{n} | N_{jk} - N'_{jk} || M'_{kn} | r^{j-n}, \\ &\leqslant O(1) \sum_{k=j}^{n} | N_{jk} - N'_{jk} | r_1^{n-k} r^{j-n}, \\ &\leqslant O(1)(r_1/r)^{n-j} \mu_j(r_1). \end{aligned}$$

For a sequence $x \in l_1$, we have

$$\begin{aligned} |(C^{(r)}x)_j | &\leqslant \sum_{n=j}^{\infty} | C^{(r)}_{jn} || x_n |, \\ &\leqslant O(1)\, \mu_j(r_1) \sum_{n=j}^{\infty} | x_n |(r_1/r)^{n-j}, \end{aligned}$$

and therefore, for each nonnegative integer m,

$$\begin{aligned} \sum_{j=m}^{\infty} |(C^{(r)}x)_j | &\leqslant O(1)\{\max_{j\geqslant m} \mu_j(r_1)\} \sum_{j=m}^{\infty} \sum_{n=j}^{\infty} | x_n |(r_1/r)^{n-j} \\ &\leqslant O(1)\| x \|_1 \frac{r}{r - r_1} \{\max_{j\geqslant m} \mu_j(r_1)\}. \end{aligned} \tag{3.5}$$

Thus $C^{(r)}x \in l_1$. Moreover, our assumptions on $\mu_j(r_1)$, together with (3.5), imply that

$$\lim_{m\to\infty} \sum_{j=m}^{\infty} |(C^{(r)}x)_j| = 0,$$

uniformly on bounded subsets of l_1, and it follows [8] that $C^{(r)}$ is compact.

Since $C^{(r)}$ maps l_1 to l_1 it follows that C maps $\mathcal{O}_r$ to $\mathcal{O}_r$, and therefore $B = I - C$ maps $\mathcal{O}_r$ to $\mathcal{O}_r$ for all $r > R$. To show that B maps one-to-one and onto, we proceed as follows.

Since $C^{(r)}$ is compact, it will follow that $I - C^{(r)}$ maps l_1 one-to-one and onto l_1 provided that 1 is not an eigenvalue of $C^{(r)}$. The adjoint of $C^{(r)}$ is representable by the transpose $[C^{(r)}]^t$ ([17]) and the matrix $I - [C^{(r)}]^t$ is lower triangular and easily seen to be one-to-one. Thus 1 is not an eigenvalue of $[C^{(r)}]^t$ and so 1 is not an eigenvalue of $C^{(r)}$. If we let $B^{(r)} = I - C^{(r)}$, then $B^{(r)}$ maps l_1 one-to-one and onto l_1 for each $r > R$, and one can argue directly from this that B maps $\mathcal{O}_r$ one-to-one and onto $\mathcal{O}_r$ for all $r > R$. This completes the proof.

4. Proof of Theorem A

Choose a number R, $0 < R < 1$, suppose that the sequence $\{z_n\}_{n=0}^{\infty}$ satisfies

$$|z_n| \leqslant WRR_{n+1}/d_{n+1}, \qquad (n = 0, 1, 2,...)$$

and define

$$\zeta_j = z_j\, d_{j+1}/R_{j+1}, \qquad (j = 0, 2,...).$$

Let the matrices A, B, A' and B' be defined as follows: for $j \leqslant n$, let

$$A_{jn} = (R_1R_2 \cdots R_je_j/R_1R_2 \cdots R_ne_n)\, Q_{n-j}(0; z_j,..., z_{n-1}),$$

$$B_{jn} = (R_1R_2 \cdots R_je_je_{n-j}/R_1R_2 \cdots R_ne_n)\, z_j^{n-j},$$

$$A'_{jn} = Q_{n-j}(0; \zeta_j,..., \zeta_{n-1}),$$

$$B'_{jn} = e_{n-j}\zeta_j^{n-j}.$$

For $j < n$, we have

$$\sum_{k=j}^{n} B_{jk}A_{kn} = \frac{R_1 \cdots R_je_j}{R_1 \cdots R_ne_n} \sum_{k=j}^{n} z_j^{k-j}\, e_{k-j}Q_{n-k}(0; z_k,..., z_{n-1})$$

$$= \frac{R_1 \cdots R_je_j}{R_1 \cdots R_ne_n} \sum_{m=0}^{n-j} z_j^{m}e_mQ_{n-j-m}(0; z_{j+m},..., z_{n-1}).$$

By (2.6) and (2.2),

$$\sum_{k=j}^{n} B_{jk}A_{kn} = \frac{R_1 \cdots R_j e_j}{R_1 \cdots R_n e_n} Q_{n-j}(z_j\,; z_j ,\ldots, z_{n-1}) = 0,$$

and therefore $BA = I$. Similarly, $B'A' = I$. Since these matrices are all upper triangular, we also have $AB = A'B' = I$.

We wish to show that A and B satisfy (3.3) and (3.4), and this will be true provided that A' and B' satisfy the hypotheses of Theorem 3.3.

LEMMA 4.1. *Each of A' and B' map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for all $r > R$.*

Proof. If $j \leqslant n$, then

$$\begin{aligned} | B'_{jn} | R^{j-n} &\leqslant (d_{j+1} | z_j |/R_{j+1})^{n-j} e_{n-j} R^{j-n} \\ &\leqslant e_{n-j} W^{n-j} = W^{n-j}/d_1 d_2 \cdots d_{n-j} . \end{aligned}$$

Since $\{d_n\}$ is nondecreasing, (1.7) implies

$$| B'_{jn} | R^{j-n} \leqslant (W/d_1)^{n-j} \leqslant 1,$$

and thus

$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} | B'_{jn} | R^{j-n}\}^{1/n} \leqslant 1.$$

To obtain the corrresponding inequality for the matrix A', let $w_k = \zeta_k/(WR)$ and observe that (2.1), (1.6) and the conditions $| w_k | \leqslant 1$ imply

$$\begin{aligned} | A'_{jn} | R^{j-n} &= | Q_{n-j}(0; \zeta_j ,\ldots, \zeta_{n-1})| R^{j-n}, \\ &= (WR)^{n-j} | Q_{n-j}(0; w_j ,\ldots, w_{n-1})| R^{j-n}, \\ &\leqslant W^{n-j} H_{n-j} \leqslant 1. \end{aligned}$$

The desired conclusion now follows from Theorem 3.2.

As in Theorem 3.3, let

$$\mu_j(r) = \sum_{n=j}^{\infty} | B_{jn} - B'_{jn} | r^{j-n}, \qquad (r > R, j \geqslant 0)$$

We now prove

LEMMA 4.2. $| A'_{jn} | r^{j-n} = O(1)$, *for all n, j and $r > R$, and $\mu_j \to 0$ uniformly in (R, ∞).*

Proof. If $r > R$ and $j \leqslant n$, then

$$| A'_{jn} | r^{j-n} = | Q_{n-j}(0; \zeta_j, ..., \zeta_{n-1}) | r^{j-n}$$
$$\leqslant (WR)^{n-j} H_{n-j} r^{j-n} \leqslant (R/r)^{n-j} \leqslant 1,$$

and this establishes the first part of the lemma. To show that $\mu_j(r) \to 0$, observe that

$$\mu_j(r) = \sum_{n=j}^{\infty} \left| \frac{R_1 \cdots R_j e_j e_{n-j}}{R_1 \cdots R_n e_n} z_j^{n-j} - e_{n-j} \zeta_j^{n-j} \right| r^{j-n},$$

$$\leqslant \sum_{n=j}^{\infty} e_{n-j} | \zeta_j |^{n-j} r^{j-n} \left| \frac{R_1 \cdots R_j e_j}{R_1 \cdots R_n e_n} \left(\frac{R_{j+1}}{d_{j+1}} \right)^{n-j} - 1 \right|,$$

$$\leqslant \sum_{n=j}^{\infty} e_{n-j} \left(\frac{WR}{r} \right)^{n-j} \left| \frac{R_{j+1}^{n-j} e_j}{R_{j+1} \cdots R_n e_n} \frac{1}{d_{j+1}^{n-j}} - 1 \right|,$$

$$\leqslant \sum_{k=0}^{\infty} e_k W^k \left| \frac{R_{j+1}^k e_j}{R_{j+1} \cdots R_{j+k} e_{j+k} d_{j+1}^k} - 1 \right|. \tag{4.1}$$

We will show that the hand side of (4.1) tends to 0 with j. To facilitate this, we introduce the sequence of functions

$$\varphi_n(z) = \sum_{k=0}^{\infty} \frac{e_k e_n}{e_{n+k} d_{n+1}^k} z^k, \qquad n = 0, 1, 2,....$$

Since $\{d_n\}$ is nondecreasing,

$$\varphi_n(| z |) \leqslant e_n \sum_{k=0}^{\infty} (d_{n+k})^n \frac{| z |^k}{d_{n+1}^k},$$

and it follows that $\varphi_n(z)$ has radius of convergence at least d_{n+1}. Writing

$$e_n / e_{n+k} d_{n+1}^k = d_{n+1} \cdots d_{n+k} / d_{n+1}^k$$

and using the fact that $\{d_{n+1}/d_n\}$ is nonincreasing, it is easy to show that the sequence $\{\varphi_n(| z |)\}_{n=k}^{\infty}$ is nonincreasing for each z in $| z | < d_{k+1}$. Let $\epsilon > 0$ and choose r_0 so that $W < r_0 < d_1$. Let N be a positive integer such that

$$(W/r_0)^N \varphi_N(r_0) < \epsilon/4.$$

Choose an integer $N_1 \geqslant N$ such that $j \geqslant N_1$ implies that

$$\sum_{k=0}^{N-1} W^k e_k \left| \frac{R_{j+1}^k e_j}{R_{j+1} \cdots R_{j+k} e_{j+k} d_{j+1}^k} - 1 \right| < \epsilon/2. \tag{4.2}$$

For $j \geqslant N_1$, we have

$$\sum_{k=N}^{\infty} \frac{R_{j+1}^k e_j e_k W^k}{R_{j+1} \cdots R_{j+k} e_{j+k} d_{j+1}^k} + \sum_{k=N}^{\infty} e_k W^k \leqslant \sum_{k=N}^{\infty} \frac{e_j e_k}{e_{j+k} d_{j+1}^k} W^k + \sum_{k=N}^{\infty} e_k W^k,$$

$$\leqslant 2 \sum_{k=N}^{\infty} \frac{e_j e_k}{e_{j+k} d_{j+1}^k} W^k,$$

$$= 2 \left(\frac{W}{r_0}\right)^N \sum_{k=N}^{\infty} \frac{e_j e_k}{e_{j+k} d_{j+1}^k} \left(\frac{W}{r_0}\right)^{k-N} r_0^k,$$

$$\leqslant 2 \left(\frac{W}{r_0}\right)^N \varphi_j(r_0),$$

$$\leqslant 2 \left(\frac{W}{r_0}\right)^N \varphi_N(r_0) < \epsilon/2.$$

In view of (4.2), $j \geqslant N_1$ implies $\mu_j(r) < \epsilon$, and this completes the proof.

Collecting the results of Lemmas 4.1 and 4.2, we now have

$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} | A_{jn} | R^{j-n}\}^{1/n} \leqslant 1,$$
$$\limsup_{n\to\infty} \{\max_{0\leqslant j\leqslant n} | B_{jn} | R^{j-n}\}^{1/n} \leqslant 1. \tag{4.3}$$

Proof of Theorem A. If k is a positive integer, (2.7) implies

$$\left| f(z) - \sum_{n=k}^{k-1} \mathscr{D}^n f(z_n)\, Q_n(z; z_0, \ldots, z_{n-1}) \right|$$

$$\leqslant \sum_{n=k}^{\infty} | \mathscr{D}^n f(0)| | Q_n(z; z_0, \ldots, z_{k-1}, 0, \ldots, 0)|. \tag{4.4}$$

Suppose first that $c(E) = \infty$; this implies that f is entire. Let $\epsilon > 0$ such that $R(1+\epsilon)^2 < 1$. Since $\tau_E(f) \leqslant 1$, then

$$| \mathscr{D}^n f(0)| R_1 \cdots R_n e_n \leqslant O(1)(1+\epsilon)^n$$

for all n. By (4.3),

$$| Q_{j-m}(0; z_m, \ldots, z_{j-1})| \leqslant O(1)(1+\epsilon)^j R^{j-m} R_1 \cdots R_j e_j/(R_1 \cdots R_m e_m)$$

for all j and m. Equation (2.5) therefore implies

$$| Q_j(z; z_0, \ldots, z_{j-1})| \leqslant O(1)(1+\epsilon)^j R_1 \cdots R_j e_j R^j \sum_{m=0}^{j} \frac{| z |^m}{R^m R_1 \cdots R_m}$$

$$\leqslant O(1)(1+\epsilon)^j R_1 \cdots R_j e_j R^j E(| z |/R),$$

and this yields

$$| Q_n(z; z_0, \dots, z_{k-1}, 0, \dots, 0)| = \left| e_n z^n - \sum_{j=0}^{k-1} e_{n-j} z_j^{n-j} Q_j(z; z_0, \dots, z_{j-1}) \right|$$

$$\leqslant e_n | z |^n + O(1)\, E(| z |/R) \sum_{j=0}^{k-1} e_{n-j} | z_j |^{n-j} (1 + \epsilon)^j R_1 \cdots R_j e_j R^j.$$

Thus

$$\sum_{n=k}^{\infty} | \mathscr{D}^n f(0)| \, | Q_n(z; z_0, \dots, z_{k-1}, 0, \dots, 0)| \leqslant \sum_{n=k}^{\infty} | \mathscr{D}^n f(0)| \, e_n | z |^n$$

$$+ O(1)\, E(| z |/R) \sum_{n=k}^{\infty} | \mathscr{D}^n f(0)| \sum_{j=0}^{k-1} e_{n-j} | z_j |^{n-j} e_j (1 + \epsilon)^j R_1 \cdots R_j R^j.$$

Using (1.4) and the bound on $| z_j |$, the right-hand side of (4.4) does not exceed

$$\sum_{n=k}^{\infty} | a_n | \, | z |^n + O(1)\, E(| z |/R) \sum_{n=k}^{\infty} | \mathscr{D}^n f(0)| \, e_n R_1 \cdots R_n R^n \sum_{j=0}^{k-1}$$

$$\times \left[\frac{e_{n-j} e_j d_1^{n-j}}{e_n d_{j+1}^{n-j}} \frac{R_1 \cdots R_j R_{j+1}^{n-j}}{R_1 \cdots R_n} \times \left(\frac{W}{d_1} \right)^{n-j} (1 + \epsilon)^j \right],$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. From the fact that $\{d_{n+1}/d_n\}$ is nonincreasing, it is not difficult to show that

$$e_{n-j} e_j d_1^{n-j} / e_n d_{j+1}^{n-j} \leqslant 1.$$

Since $\{R_n\}$ is nondecreasing, we have

$$R_1 \cdots R_j R_{j+1}^{n-j} / R_1 \cdots R_n \leqslant 1.$$

From these estimates and (1.7), (4, 4) does not exceed

$$\sum_{n=k}^{\infty} | a_n | \, | z |^n + O(1)\, E(| z |/R) \sum_{n=k}^{\infty} n[R(1 + \epsilon)^2]^n.$$

Therefore, (4.4) tends uniformly to 0 on bounded parts of the plane.

Suppose now that $c(E) < \infty$. Since $\tau_E(f) = c(E)/c(f) \leqslant 1$, it suffices to show that the right-hand side of (4.4) converges to 0 uniformly on compact subsets of the annulus $Rc(E) \leqslant | z | < c(f)$. From (2.6) and (4.3),

$$| Q_j(z; z_0, \dots, z_{j-1})| \leqslant O(1)\, R_1 \cdots R_j e_j (1 + \epsilon)^j \sum_{m=0}^{j} \frac{R^{j-m} | z |^m}{R_1 \cdots R_m}.$$

Since $c(E) = \sup_n R_n$, then

$$1/(R_1 \cdots R_m) = (R_{m+1} \cdots R_j)/(R_1 \cdots R_j) \leqslant [c(E)]^{j-m}/(R_1 \cdots R_j)$$

for $m \leqslant j$. For $Rc(E) \leqslant |z| < c(f)$, we therefore have

$$| Q_j(z; z_0, \ldots, z_{j-1})| \leqslant O(1)|z|^j e_j(1+\epsilon)^j \sum_{m=0}^{j} (Rc(E)/|z|)^{j-m},$$

$$\leqslant O(1)|z|^j e_j(1+\epsilon)^j (j+1).$$

As in (4.5), this shows that the right-hand side of (4.4) does not exceed

$$\sum_{n=k}^{\infty} |a_n||z|^n + O(1) \sum_{n=k}^{\infty} R^n |a_n| \sum_{j=0}^{k-1} R_{j+1}^n (j+1)(1+\epsilon)^j \frac{|z|^j}{R_{j+1}^j R^j}.$$

Since $c(E) \geqslant R_{j+1}$, we have $|z|^j/R_{j+1}^j R^j \geqslant 1$. Therefore (4.4) is less than

$$\sum_{n=k}^{\infty} |a_n||z|^n + O(1) \sum_{n=k}^{\infty} |a_n||z|^n \sum_{j=0}^{k-1} (j+1)(1+\epsilon)^j$$

$$\sum_{n=k}^{\infty} |a_n||z|^n + O(1) \sum_{n=k}^{\infty} (n+1)^2 |a_n| [(1+\epsilon)|z|]^n.$$

Since this last expression can be made arbitrarily small on compact subsets of $Rc(E) \leqslant |z| < c(f)$, the proof of Theorem A is complete.

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